First-order perturbed Korteweg−de Vries solitons

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We consider the Korteweg–de Vries equation with a perturbation arising naturally in many physical situations. Although being asymptotically integrable, we show that the corresponding perturbed solitons do not have the usual scattering properties. Specifically, we show that there is a solution, correct up to $O(\epsilon)$, where ϵ is the perturbative parameter, consisting, at $t \to -\infty$, of two superposed deformed solitons characterized by wave numbers k_1 and k_2 that give rise, for $t \to +\infty$, to the same but phase-shifted superposed solitons plus a coupling term depending on k_1 and k_2. We also find the condition on the original equation for which this coupling vanishes. [S1063-651X(98)06004-8]

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The importance of certain *integrable partial differential equations* for the description of physical systems comes from the fact that they may describe asymptotic limits of equations supposed to govern real systems. The property of integrability, although being rare in general, in the realm of asymptotic equations is ubiquitous [1]. A notorious example is that of the Korteweg–de Vries (KdV) equation. This equation describes the long-wavelength limit of spatially extended systems that are conservative and dispersive. The nonlinear Schrödinger (NLS) equation also enjoys this universality character, describing the slow nonlinear modulation of wave amplitude in dispersive systems. The physical systems concerned form a wide range of examples, including, to quote a few [2], water waves, gas dynamics, plasma physics and waves in ferrites. The concept of solitons is directly associated with equations that are integrable by the inverse scattering method [3]. Indeed, such equations display N-soliton solutions and a general solution for sufficiently well-behaved initial conditions evolves asymptotically in time to a superposition of solitons and radiative terms.

Predictions derived from equations such as the KdV and NLS equations, which are obtained from perturbation theory, must pass the test of being stable against higher-order perturbations. To date, a plethora of results exists concerning the behavior of solutions of perturbed equations, most frequently obtained through the use of the perturbed inverse scattering transform method [4]. To make our point, we first specialize the problem. We will be considering perturbations to the KdV equation, thus equations reading

$$u_t = 6uu_x - u_{xxx} + \epsilon P(u), \qquad (1)$$

where u is a function of (x,t), $P(u)$ is a function of u and its derivatives in x, $\epsilon \ll 1$ is a perturbative parameter, and subscripts denote differentiation. We will not be concerned with an arbitrary $P(u)$. Rather, we will be concerned with what could be called *natural* perturbation, namely, those $P(u)$ arising in perturbative expansions from basic equations that are conservative and dispersive, with a dispersion relation admitting an expansion of the form $\omega(k) = a_1 k + a_3 k^3 + a_5 k^5 + \cdots$. We further assume that no constants scale with ϵ. To have a better understanding of what is meant by *natural*, consider a conservative and dispersive system and look

for its long-wavelength, weak, nonlinear limit, by introducing slow space and time variables $\xi = \epsilon(x - ct)$, $\tau = \epsilon^3 t$, where $\epsilon \ll 1$. This results in an equation of the form

$$u_t = P_0[u] + \epsilon P_1[u] + \epsilon^2 P_2[u] + \cdots,$$

where $P_i[u]$ are polynomials in u and its ξ derivatives that scale homogeneously when $u \sim \epsilon^{-2}$, $\partial_\xi \sim \epsilon^{-1}$, and $\partial_\tau \sim \epsilon^{-3}$. The scaling order must be such that $\epsilon^n P_n$ scales like u_τ. This restricts the possible forms of P_i. For instance, P_0 is forced to contain only terms proportional to $u_{\xi\xi\xi}$ and uu_ξ, resulting in the KdV equation. P_1 is what is called a natural perturbation. It contains all the possible terms allowed by the scaling and it represents a generic perturbation. It is not difficult to see that it must be of the form

$$P(u) = \alpha_0 u_{xxxxx} - 10\alpha_1 uu_{xxx} - 20\alpha_2 u_x u_{xx} + 30\alpha_3 u^2 u_x, \qquad (2)$$

where α_i are arbitrary constants. One could ask the following question: When is an ''unnatural'' perturbation allowed? In the context of conservative and dispersive equations the answer is that we would need a physical constant to scale with ϵ. The definition of naturalness is meant to exclude this case. Notice that the same hypothesis is assumed when we speak of the universality of the KdV equation, which would not represent a universal limit of dispersive systems if arbitrary scaling of constants were allowed.

If in Eq. (2) $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3$, then the resulting equation obtained by inserting Eq. (2) into Eq. (1) is integrable and displays N-soliton solutions. For instance, the one-soliton solution reads simply

$$u = -2k \operatorname{sech}^2\{k[x - (4k^2 - 16\alpha_0 \epsilon k^4)t]\}. \qquad (3)$$

Let us now ask the central question: Given Eq. (1), with the perturbation (2), can we still define the concept of solitons? Explicitly, do we still have a solution that for $t \to \pm\infty$ is the superposition of functions $\phi_i(x,t,k_i)$, each of them depending only on one k_i, and such that they scatter elastically, differing from $t \to -\infty$ to $t \to +\infty$ by a phase shift only? It should be remarked that we are working with an equation coming from a perturbative expansion, truncated at $O(\epsilon)$. Thus, we do not look for exact solutions, but for solutions

correct up to $O(\epsilon)$. Thus we arrive at the formalism of asymptotic integrability, as introduced in [5–7].

Suppose that we make an ϵ-dependent transformation $u \to w$ of the form

$$u = w + \epsilon \phi(w), \qquad (4)$$

known as a near-identity transformation. What has been shown in [5,7] is that Eq. (1), with $P(u)$ given by Eq. (2), can always be transformed to

$$w_t = 6ww_x - w_{xxx} + \alpha_0 \epsilon (w_{xxxxx} - 10ww_{xxx} - 20w_xw_{xx}$$
$$+ 30w^2w_x) + O(\epsilon^2), \qquad (5)$$

which is just the sum of the KdV equation and the first higher-order equation in the KdV hierarchy [3], and its solutions are the solutions of the KdV equation with a renormalized velocity, just like in Eq. (3). Thus, in order to find solutions of the original problem we have to insert the solutions of Eq. (5) into Eq. (4). Before giving the explicit form of the near-identity transformation we should stress that it not possible, in general, to extend this procedure to arbitrarily higher orders. This comes from the existence of obstacles to asymptotic integrability as discussed in [6] for the NLS equation, but equally applicable to the KdV equation. Furthermore, it is possible to find a transformation that maps our original equation (1) to the KdV equation itself but that is x dependent [7]. However, this is not of practical use here.

The correct form of $\phi(w)$ turns out to be

$$\phi(w) = \alpha w^2 + \beta w_{xx} + \gamma w_x \partial^{-1} w, \qquad (6)$$

where ∂^{-1} denotes integration with respect to x and α, β, and γ are given in terms of α_i by

$$\alpha = \frac{1}{3}(10\alpha_0 + 5\alpha_1 - 15\alpha_3), \qquad (7a)$$

$$\beta = \frac{1}{6}(15\alpha_3 - 10\alpha_2 - 5\alpha_0), \qquad (7b)$$

$$\gamma = \frac{10}{3}(\alpha_0 - \alpha_1). \qquad (7c)$$

We can now look for the u coming from the one-soliton and two-soliton solutions, for example. This is instructive for the one-soliton case, but the expression for the two-soliton case is not very illuminating. For this reason, we should look at the asymptotics for $t \to \pm\infty$. This is most easily accomplished by using well-known techniques of soliton theory. First, transform from w to F by

$$w = -2(\ln F)_{xx}, \qquad (8)$$

thus obtaining a transformed $\phi(F)$:

$$\phi = \frac{4}{F^4}\left[\left(\alpha + \frac{3\beta}{2}\right)F^2F_{xx}^2 - (2\alpha + 6\beta + 3\gamma)FF_x^2F_{xx}\right.$$
$$\left. + (\alpha + 3\beta + 2\gamma)F_x^4 + (2\beta + \gamma)F^2F_xF_{xxx} - \left(\frac{\beta}{2}\right)F^3F_{xxxx}\right].$$
$$(9)$$

As is well known, if we define $\eta_k = k[x - (k^2 - \epsilon\alpha_0 k^4)t]$; then, for a one-soliton solution we have

$$F = 1 + \exp(\eta_k) \qquad (10)$$

and for a two-soliton solution

$$F = 1 + \exp(\eta_{k_1}) + \exp(\eta_{k_2}) + \exp(\eta_{k_1} + \eta_{k_2} + A_{12}), \qquad (11)$$

with $\exp A_{12} = [(k_1 - k_2)/(k_1 + k_2)]^2$. Inserting expression (10) into Eq. (9) and going through the algebra gives us the perturbation to the one-soliton solution:

$$\frac{\phi}{k^4} = -\left(\frac{\beta + \gamma}{2}\right)\text{sech}^2(\eta_k/2) + \left(\frac{\alpha + 3\beta + 2\gamma}{4}\right)\text{sech}^4(\eta_k/2)$$
$$- \left(\frac{\gamma}{2}\right)\text{sech}^2(\eta_k/2)\tanh(\eta_k/2). \qquad (12)$$

The first term represents a perturbation to the amplitude of the solitary wave. The second and third terms represent deformations of the soliton form. However, in this case the term $\text{sech}^2(\eta_k/2)\tanh(\eta_k/2)$ can always be eliminated. Indeed, in Eq. (4) a term Aw_x can always be added, A being an arbitrary constant that can be adjusted to eliminate the third term in Eq. (12). This is related to the fact that, if we had tried to solve Eq. (1) by a perturbative series, at $O(\epsilon)$ we could always sum a solution of the linearized KdV equation. Without this third term, we have the result of [8]. Let us now go to the two-soliton case, but let us take into account the existence of an additional term Aw_x from the beginning. Instead of writing out the solution explicitly, we give the asymptotic behavior for $t \to \pm\infty$, taking the limits by using standard techniques [3]. We obtain the following results. First, define

$$\psi(\eta_k, k) = k^4\left[-\left(\frac{\beta + \gamma}{2}\right)\text{sech}^2(\eta_k/2)\right.$$
$$\left. + \left(\frac{\alpha + 3\beta + 2\gamma}{4}\right)\text{sech}^4(\eta_k/2)\right]. \qquad (13)$$

With this definition, we have, for $t \to +\infty$,

$$\phi = \psi(\eta_{k_1}, k_1) + \psi(\eta_{k_2} + A_{12}, k_2) + \left(\frac{Ak_1^3\gamma}{2} - \frac{k_1^4}{2}\gamma\right)$$
$$\times \text{sech}^2(\eta_{k_1}/2)\tanh(\eta_{k_1}/2)$$
$$+ \left(\frac{\gamma Ak_2^3}{2} - \gamma k_1 k_2^3 + \gamma \frac{k_2^4}{2}\right)\text{sech}^2[(\eta_{k_2} + A_{12})/2]$$
$$\times \tanh[(\eta_{k_2} + A_{12})/2], \qquad (14)$$

and for $t \to -\infty$,

$$\phi = \psi(\eta_{k_1} + A_{12}, k_1) + \psi(\eta_{k_2}, k_2) + \left(\frac{A k_2^3 \gamma}{2} - \frac{k_2^4}{2} \gamma \right)$$

$$\times \mathrm{sech}^2(\eta_{k_2}/2) \tanh(\eta_{k_2}/2) + \left(\frac{\gamma A k_1^3}{2} - \gamma k_2 k_1^3 + \gamma \frac{k_1^4}{2} \right)$$

$$\times \mathrm{sech}^2[(\eta_{k_1} + A_{12})/2] \tanh[(\eta_{k_1} + A_{12})/2]. \qquad (15)$$

If we now impose that for $t \to -\infty$ we have a sum of functions, each one depending only on either k_1 or k_2, we must choose $A = 2k_2$. However, with this choice, for $t \to +\infty$ we will get a coupling term involving k_1 and k_2. Indeed, we can see that there is no constant A such that for both $t \to \pm\infty$ we can have a superposition of functions, none of which depends on the mixed (k_1, k_2) terms. We see thus that we do not have a process of scattering of two solitons that are uncorrelated asymptotically in time. Even for a solution consisting of the superposition of two one solitons at $t \to -\infty$, a linkage appears for $t \to +\infty$ through a (k_1, k_2) coupling. This means that we cannot define two separate particlelike objects that scatter and emerge maintaining their identity. Note,

however, that if $\gamma = 0$, then this linkage disappears and the usual picture of elastic scattering of solitons, although deformed, shows up.

Let us now summarize our results. We have a perturbed KdV equation, where the perturbation is, generically, nonintegrable. Instead of looking for an exact solution, we search for an expression valid up to $O(\epsilon)$, as the original equation is supposed to be valid only to this same order. In this sense, we have found the general effects of the perturbation given by Eq. (2), which are summarized in Eqs. (14) and (15). Qualitatively we can say the following: In the general case, the solitons are asymptotically deformed and do not preserve the usual particlelike properties because of the appearance of a linking term involving k_1 and k_2. However, an interesting case emerges when $\alpha_0 = \alpha_1$ in Eq. (2), in which there are particlelike structures, deformed solitons, that have asymptotically, up to $O(\epsilon)$, the same collisional properties as solitons in integrable systems.

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